Packaging Theories of Higher Order Logic

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Talk Plan

1. Introduction
2. Combining Theories
3. Packaging Theories
4. Implementation Notes
5. Summary
Interactive theorem proving is growing up.
It has moved beyond toy examples of mathematics and program verification.

- The FlySpeck project is driving the HOL Light theorem prover towards a formal proof of the Kepler sphere-packing conjecture.
- The CompCert project used the Coq theorem prover to verify an optimizing compiler from a large subset of C to PowerPC assembly code.

There is a need for theory engineering techniques to support these major verification efforts.
- Theory engineering is to proving as software engineering is to programming. “Proving in the large.”
The OpenTheory project aims to apply software engineering principles to the development of higher order logic theories.\(^1\)

The initial case study for the project is Church’s simple theory of types, extended with Hindley-Milner style type variables.

- The logic implemented by HOL4, HOL Light and ProofPower.

By focusing on a concrete case study we aim to investigate the issues surrounding:

- Designing theory languages portable across theorem prover implementations.
- Uploading, installing and upgrading theory packages from online repositories.
- Discovering design techniques for reusable theories.
- Building a standard library of higher order logic theories.

\(^1\)OpenTheory was started in 2004 with Rob Arthan.
A theory $\Gamma \vdash \Delta$ of higher order logic consists of:

1. A set $\Gamma$ of assumption sequents.
2. A set $\Delta$ of theorem sequents.
3. A formal proof that the theorems in $\Delta$ logically derive from the assumptions in $\Gamma$.

Theories can be directly represented as OpenTheory article files, a format designed to simplify theory import and export for theorem prover implementations.

This talk will present a language for building up from article files to theory packages.
Connecting Theories

- Note that both the input assumptions and output theorems of a theory are sequent sets.
- We can therefore connect the output theorems of one theory to satisfy the input assumptions of another:

\[
A \cup B \cup C_{\text{IN}} \vdash S \cup C_{\text{OUT}}.
\]
Theory Interpretations

A theory $\Gamma \vdash \Delta$ can be applied in any context where the assumptions $\Gamma$ hold. This is called theory interpretation.

Example: The theory

$$\{id = \lambda x. \ x\} \vdash \{\forall x. \ id \ x = x\}$$

can be applied in any context with a constant $id$ having the assumed property.

Constants and type operators can be consistently renamed

$$(\Gamma \vdash \Delta)\sigma = \Gamma\sigma \vdash \Delta\sigma$$

allowing theories to be applied in even more contexts.
What Can Go Wrong?

- When connecting together theories, the connection graph must not contain any loops!
  - Theories are representations of proofs, which are directed *acyclic* graphs.
  - In this way proofs are more like combinational circuits than programs.
- A set of theorems must not have incompatible definitions for the same constant or type operator.
  - Example: The two theories

\[
\emptyset \vdash \{ c = 0 \} \quad \text{and} \quad \emptyset \vdash \{ c = 1 \}
\]

are individually fine, but must never be imported into the same context.
A Language for Theories

The following theory language allows article files and theory packages to be combined into a new theory:

```
theory ← article "filename";
| { theory* }
| local theory in theory
| interpret { interpretation* } in theory
| import package-instance;
```

Incompatible definition clashes are prevented by:
- Limiting the scope of contexts using the `local` construct.
- Renaming constant and type operators using `interpret` blocks.
Theory Package Instances

- An imported *package-instance* refers to a required theory package, specified as a *package-instance-spec*:

  \[
  \text{package-instance-spec} \leftarrow \text{require package-instance}\quad\{
  \text{import: package-instance}\quad^\ast
  \\
  \text{interpret: interpretation}\quad^\ast
  \\
  \text{package: package-name}
  \}
  \]

- A list of *package-instance-specs* specify a connection graph between theory packages.

- Each *package-instance-spec* may only import earlier *package-instance-specs*, to ensure the absence of loops.
We can now define the grammar for theory packages:

\[
\text{package} \leftarrow \text{tag}^* \\
\text{package-instance-spec}^* \\
\text{theory} \{ \text{theory} \}
\]

Tags are package meta-data:

\[
\text{tag} \leftarrow \text{name: value}
\]
Theory Package Example


name: hol-light-trivia-one-def
version: 2009.8.24
description: HOL Light definition of the unit type.

theory { article "trivia-one-def.art"; }
Theory Package Example


input-types: -> bool
input-consts: ! \(\land\) = ? T select
assumed:
  |- T
  {.} |- (!) P
  {.} |- (?) P
  {..} |- p \(\land\) q
  |- t = (t = T)
  |- (?) = \(\forall\) P. P ((select) P)
defined-types: unit
defined-consts: one one_ABS one_REP
thms:
  |- ?b. b
  |- one = select x. T
  |- (!a. one_ABS (one_REP a) = a) \(\land\)
    !r. r = (one_REP (one_ABS r) = r)
Theory Package Design

- Well-designed theory packages have:
  - a clear topic (e.g., trigonometric functions);
  - a simple set of assumptions (i.e., satisfied by standard packages);
  - a carefully chosen set of theorems (no junk, and a minimal interface if the package makes definitions);
  - and it should go without saying: no axioms!

- **Theory Engineering Challenge:** Construct a standard library of well-designed theory packages, available to all the theorem prover implementations.
Theory Package Example II

Theory Package (unit-def-1.0)

name: unit-def
version: 1.0
description: Definition of the unit type

require hol-light-thm {
}

require hol-light-trivia-one-def {
  import: hol-light-thm
}

require hol-light-trivia-one-alt {
  import: hol-light-thm
  import: hol-light-trivia-one-def
}

theory { import hol-light-trivia-one-alt; }
Theory Package Example II

Theory Package Summary (unit-def-1.0)

input-types: -> bool
input-consts: ! /
= ==? T select
assumed:
|-

|-

|-

|-

|-

|-

|-

|-

|-

defined-types: unit
defined-consts: one
thms:
|-

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Packaging Theories of Higher Order Logic
To make it easy to reason about theory package instances, we would like package instantiation to be a pure function

\[ \text{package-instance-spec} \rightarrow \Gamma \vdash \Delta. \]

Possible because the package management tool implements a purely functional logical kernel (an idea of Freek Wiedijk).

Constants and type operators contain their definitions, instead of being inserted in a symbol table, so definitions are referentially transparent:

\[
\text{let } c \equiv \text{define } \phi \text{ in } f \ c \ c \equiv (f \ (\text{define } \phi) \ (\text{define } \phi))
\]
Referential transparency means there is no difference in functionality between instantiating a theory package multiple times in the same way or instantiating it once and reusing.

However, there will likely be a big difference in performance (article files are measured in megabytes).

**Challenge:** Detecting when two `package-instance-specs` would result in the same theory.

The logical kernel similarly aims to share subterms as much as possible, in computing free variables, substitutions, etc.
This talk presented a language for combining and packaging theories.

The next challenge: build the package management infrastructure for people to contribute to building a standard library of theories.

The project web page:

\[\text{http://gilith.com/research/opentheory}\]
The concrete syntax for `package-instance-spec` evaluates to the theory

\[ \bigcup \Gamma_i \cup \left( \Gamma \sigma - \bigcup \Delta_i \right) \vdash \Delta \sigma \]

where:
- the imported `package-instance-specs` evaluate to \( \Gamma_i \vdash \Delta_i \);
- the `interpretation` rules are the renaming \( \sigma \); and
- the `package-name` is the theory \( \Gamma \vdash \Delta \).
Here is how the concrete syntax for theory is evaluated in a context with theorems $\Phi$ and renaming $\sigma$:

\[
\begin{align*}
\text{[article "[}\Gamma \vdash \Delta\text{]"]}_{\Phi,\sigma} &= \Gamma_\sigma \vdash \Phi \vdash \Delta_\sigma \\
[\{ \theta \} \}_{\Phi,\sigma} &= \emptyset \vdash \emptyset \\
[\{ \theta_1 :: \theta_2 \} \}_{\Phi,\sigma} &= \text{let } \Gamma_1 \vdash \Delta_1 = [\theta_1]_{\Phi,\sigma} \text{ in } \\
&\quad \text{let } \Gamma_2 \vdash \Delta_2 = [\{ \theta_2 \}]_{\Phi \cup \Delta_1,\sigma} \text{ in } \\
&\quad \Gamma_1 \cup \Gamma_2 \vdash \Delta_1 \cup \Delta_2 \\
[\text{local } \theta_1 \text{ in } \theta_2]_{\Phi,\sigma} &= \text{let } \Gamma_1 \vdash \Delta_1 = [\theta_1]_{\Phi,\sigma} \text{ in } \\
&\quad \text{let } \Gamma_2 \vdash \Delta_2 = [\theta_2]_{\Phi \cup \Delta_1,\sigma} \text{ in } \\
&\quad \Gamma_1 \cup \Gamma_2 \vdash \Delta_2 \\
[\text{interpret } \{ \rho \} \text{ in } \theta]_{\Phi,\sigma} &= [\theta]_{\Phi,\sigma \circ \rho} \\
[\text{import } [\Gamma \vdash \Delta]]_{\Phi,\sigma} &= \Gamma \vdash \Delta
\end{align*}
\]

Note that importing a package-instance ignores the theory context; its context is fixed by the package-instance-spec.